Differential Forms as a Supplementary Framework for Teaching Upper-Level Electricity and Magnetism

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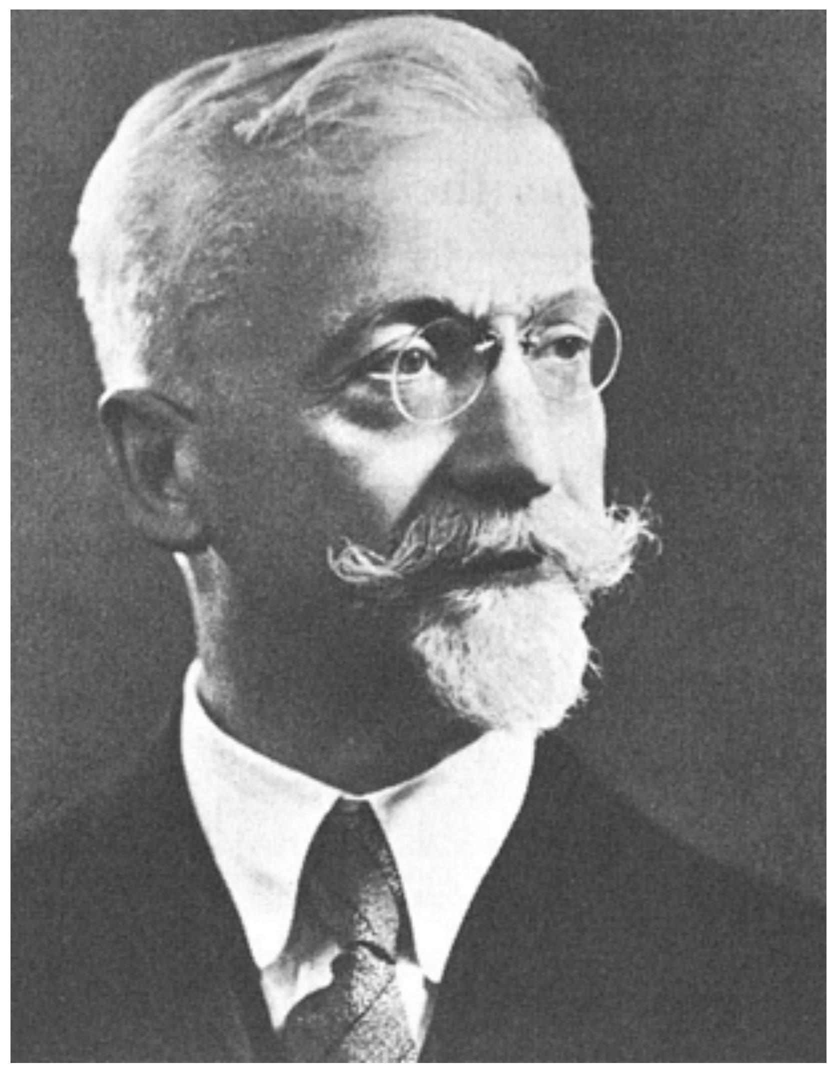
AAPT Winter Meeting 2016
New Orleans

WELLESLEY

What is the Calculus of Differential Forms?

Introduced by Élie Cartan in the early1900‘s, it is a framework for the analysis of covariant, antisymmetric tensors also called ***differential forms***.



Élie Cartan

Why Use Differential Forms in Undergraduate EM?

IEEE TRANSACTIONS ON EDUCATION, VOL. 40, NO. 1, FEBRUARY 1997 53

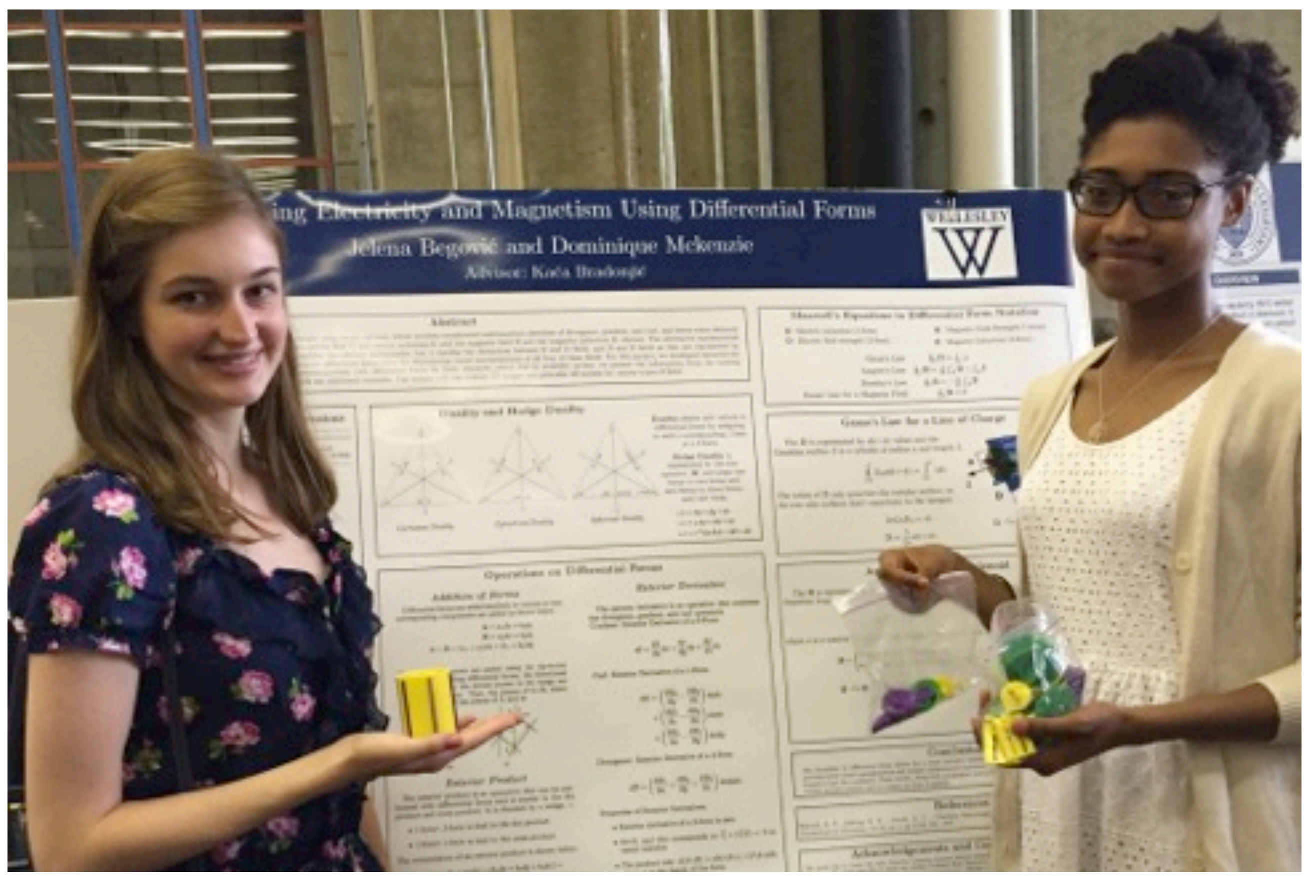
Teaching Electromagnetic Field Theory Using Differential Forms

Karl F. Warnick, Richard H. Selfridge, *Member, IEEE*, and David V. Arnold

***Abstract*— The calculus of differential forms has significant advantages over traditional methods as a tool for teaching electromagnetic (EM) field theory: First, forms clarify the relationship between field intensity and flux density, by providing distinct mathematical and graphical representations for the two types of fields. Second, Ampere's and Faraday's laws obtain graphical representations that are as intuitive as the representation of Gauss's law. Third, the vector Stokes theorem and the divergence theorem become special cases of a single relationship that is easier for the student to remember, apply, and visualize than their vector formulations. Fourth, computational simplifications result from the use of forms: derivatives are easier to employ in curvilinear coordinates, integration becomes more straightforward, and families of vector identities are replaced by algebraic rules. In this paper, EM theory and the calculus of differential forms are developed in parallel, from an elementary, conceptually oriented point of view using simple examples and intuitive motivations. We conclude that because of the power of the calculus of differential forms in conveying the fundamental concepts of EM theory, it provides an attractive and viable alternative to the use of vector analysis in teaching electromagnetic field theory.**

Wellesley College Science Center Summer Research Program

Summer 2015: Create student-friendly resources for learning about differential forms in 3 dimensions and their applications to electromagnetic theory.



Jelena Begovic (‘17) and Dominique McKenzie (‘17)

Differential Forms in 3D
A quick Tutorial

Dual to Vector Calculus

Vector Calculus	**Calculus of DF**
scalar	0-form
vector	1-form
axial vector	2-form
scalar density	3-form
dot product	wedge product
cross product	wedge product
div	exterior derivative
grad	exterior derivative
curl	exterior derivative

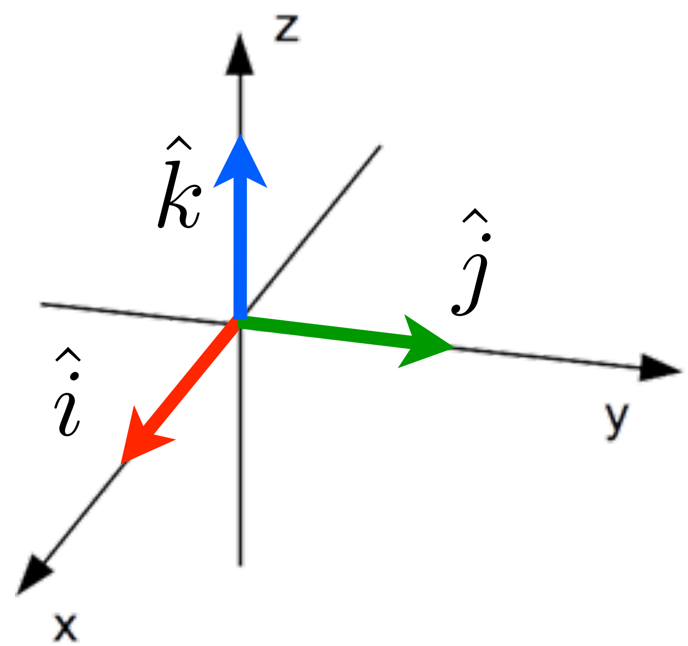
Scalars ↔ 0-forms

$$f(x, y, z)$$

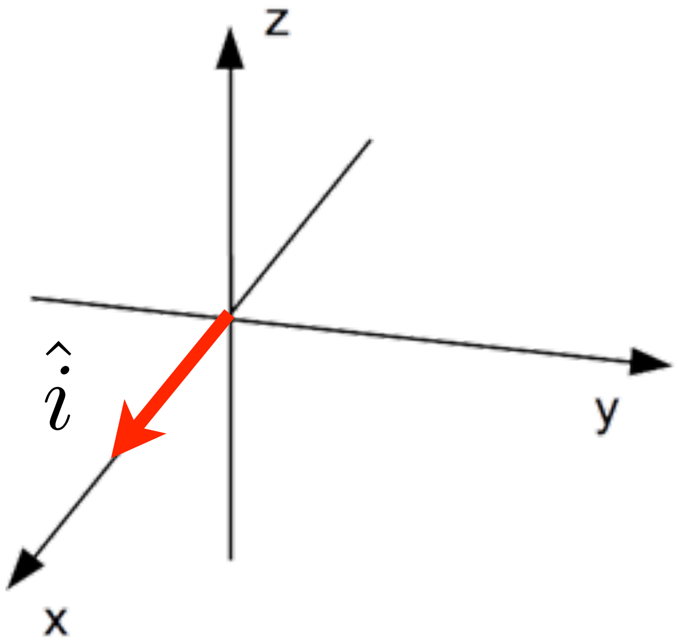
Example in 2D: Elevation

$$h(x, y)$$

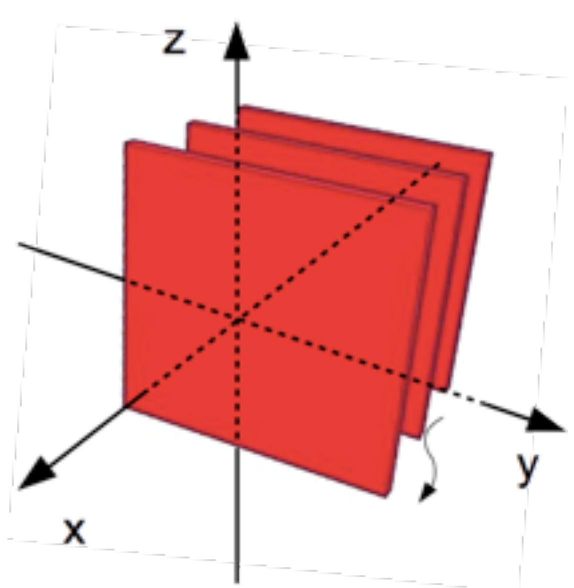
Vectors $\leftrightarrow$ 1-Forms



Vectors $\leftrightarrow$ 1-Forms

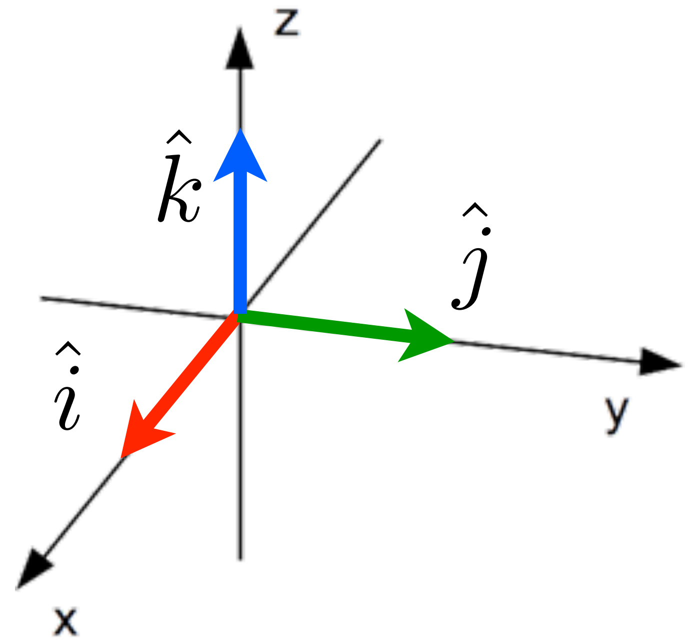


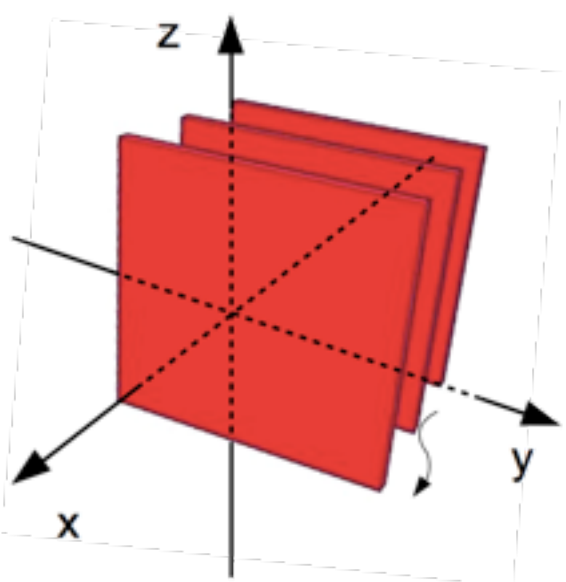
$\hat{i}$



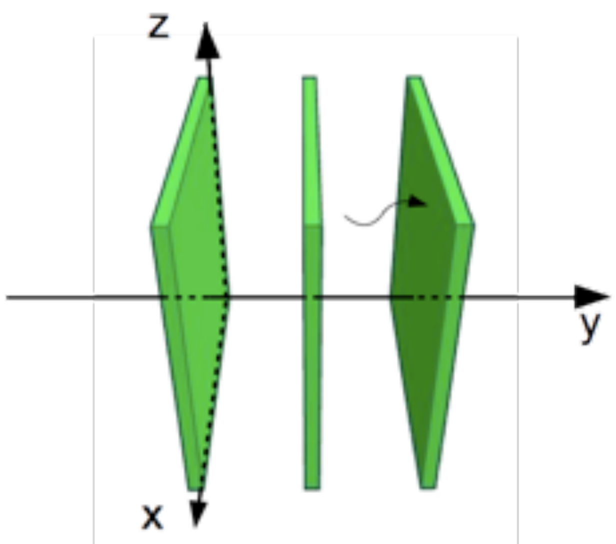
dx

Vectors $\leftrightarrow$ 1-Forms

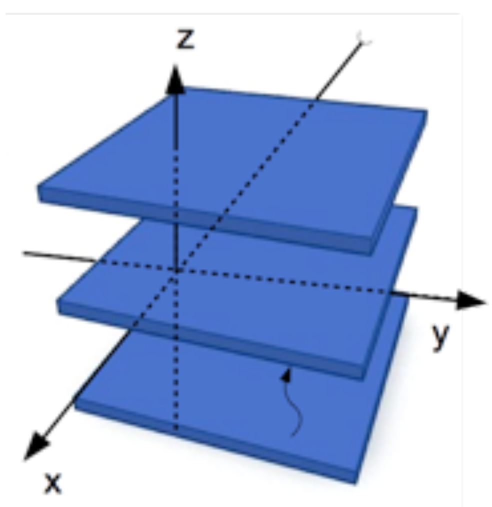




dx

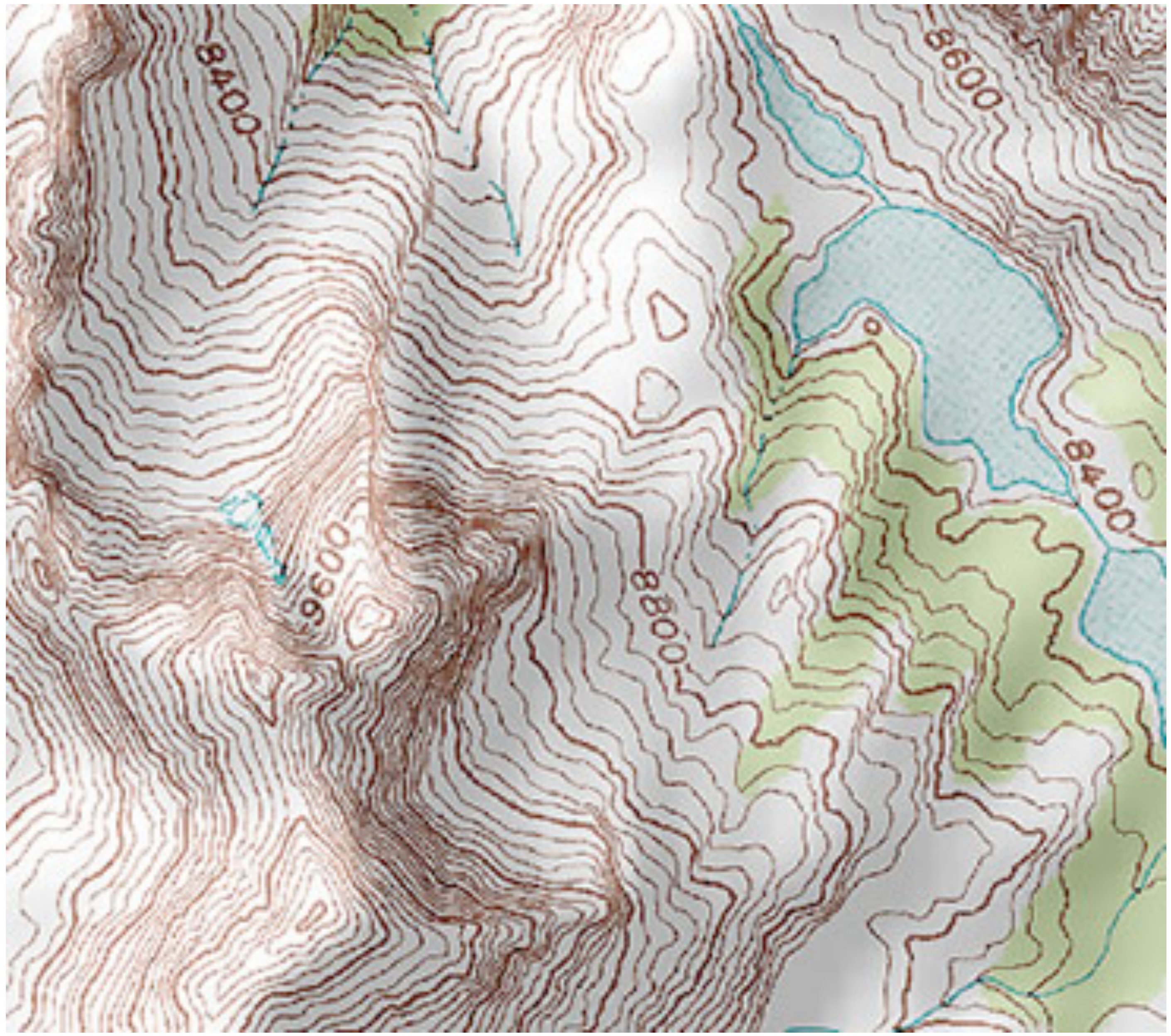


dy



dz

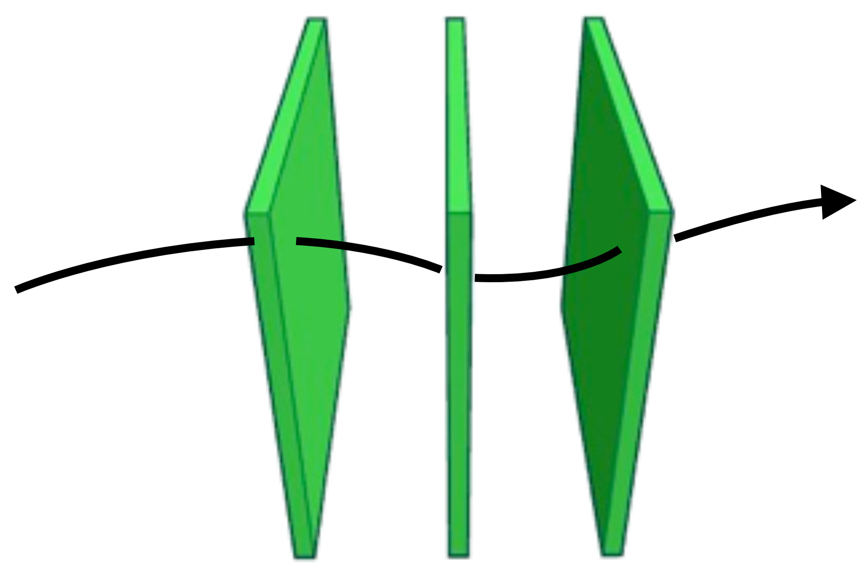
Example in 2D: Contour map



The gradient of the elevation is a 1-form.

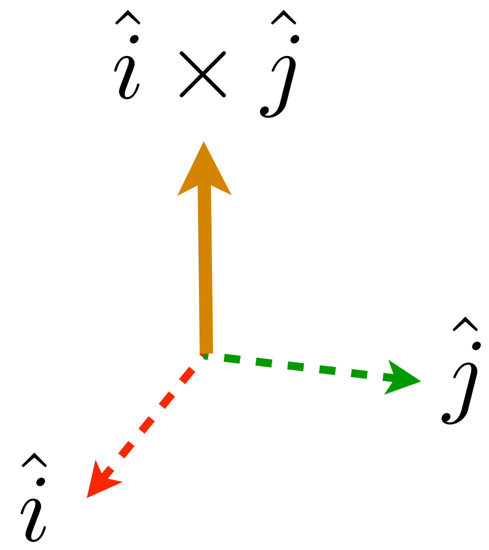
1-Forms are Integrated over Lines

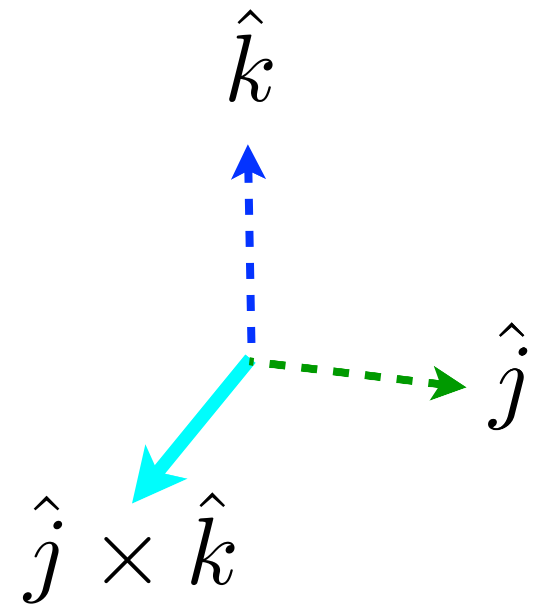
$\int_C \alpha$ can be visualized as a number of planes punctured by the path of integration.

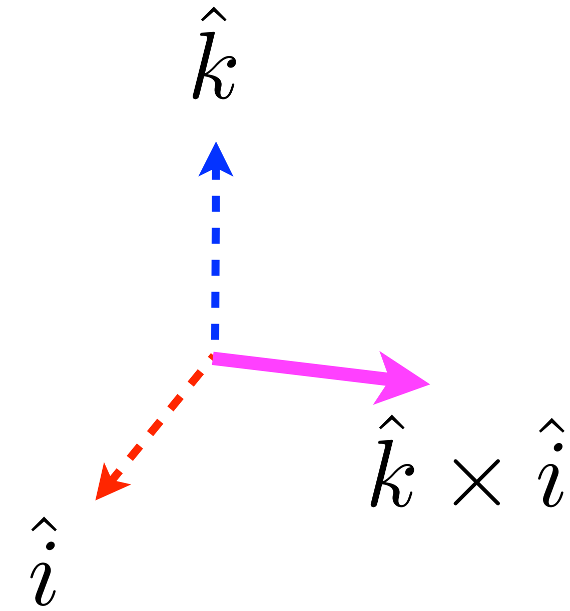


1-form α

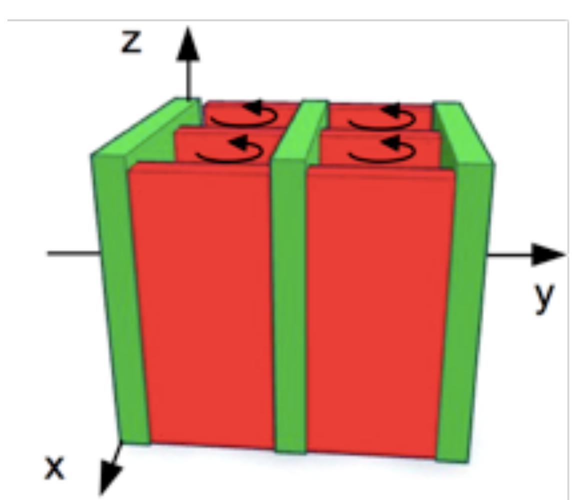
Axial Vectors $\leftrightarrow$ 2-forms



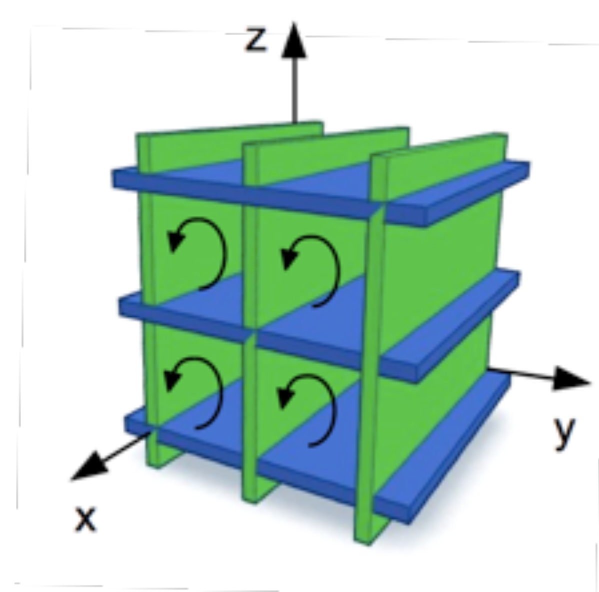




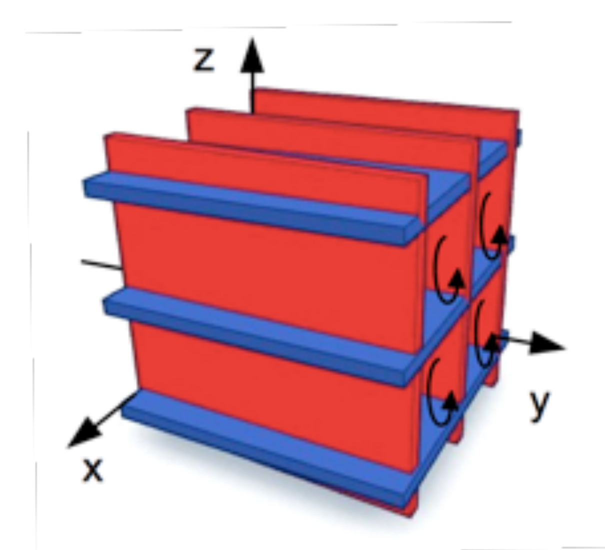
$dx \wedge dy$



$dy \wedge dz$

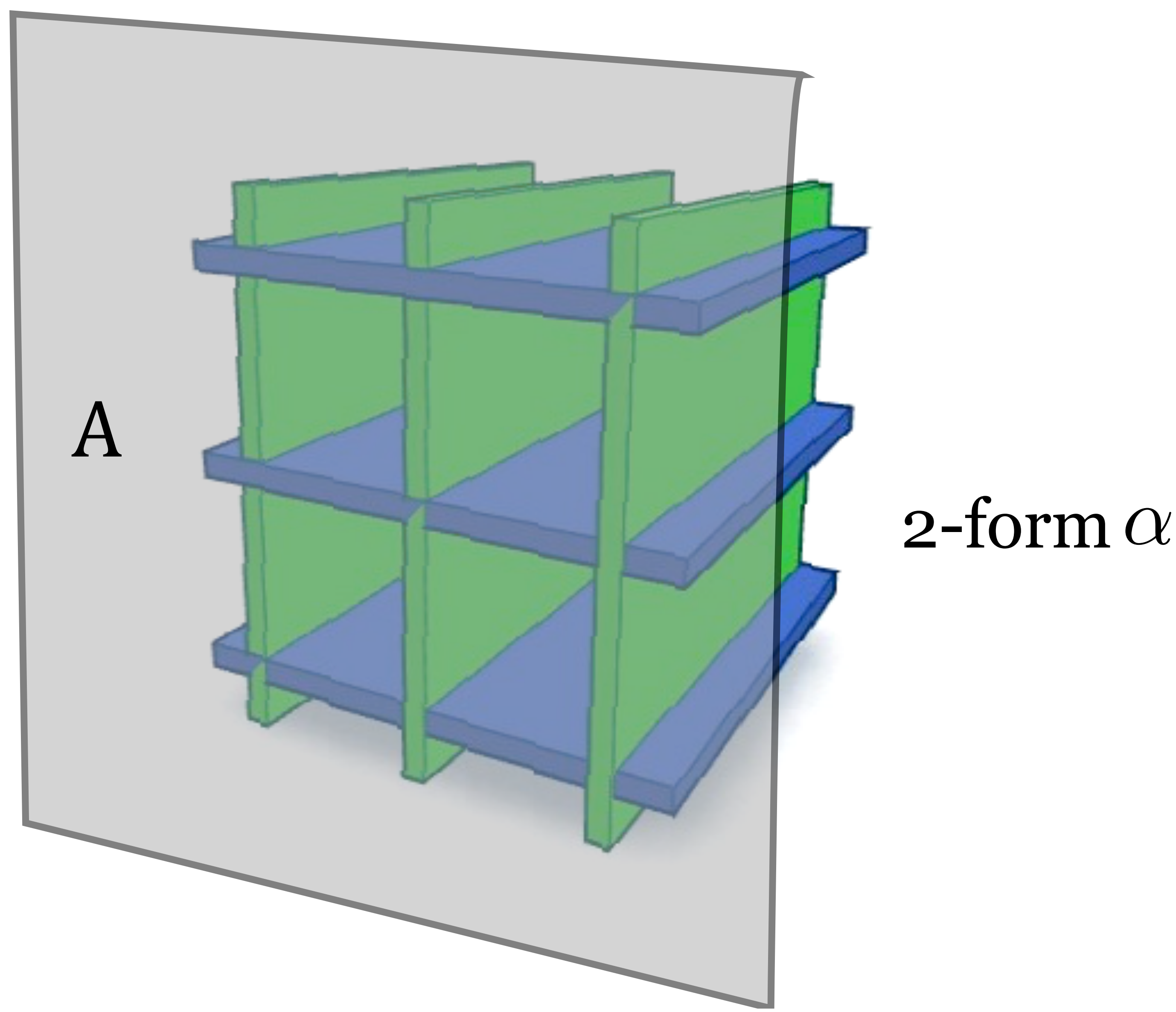


$dz \wedge dx$



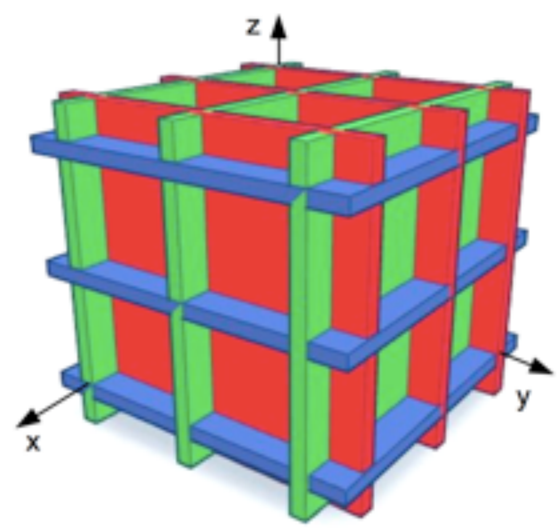
2-Forms are Integrated over Surfaces

$\int_A \alpha$ can be visualized as a number of tubes puncturing the surface of integration.



Scalar Densities ↔ 3-forms

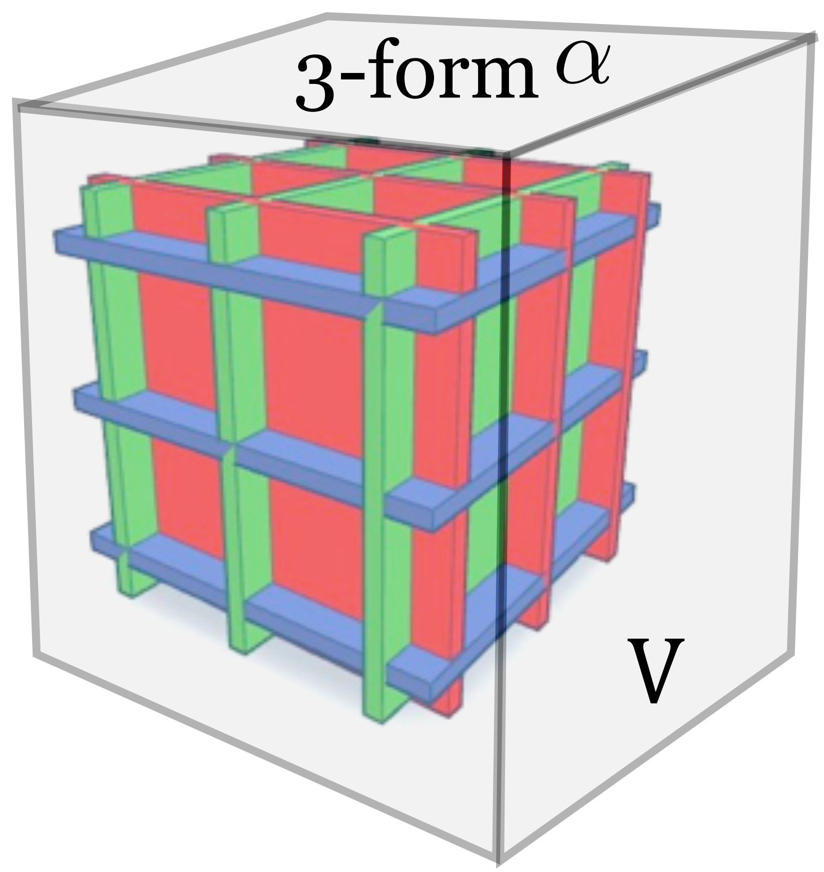
$\varrho(x,y,z)$



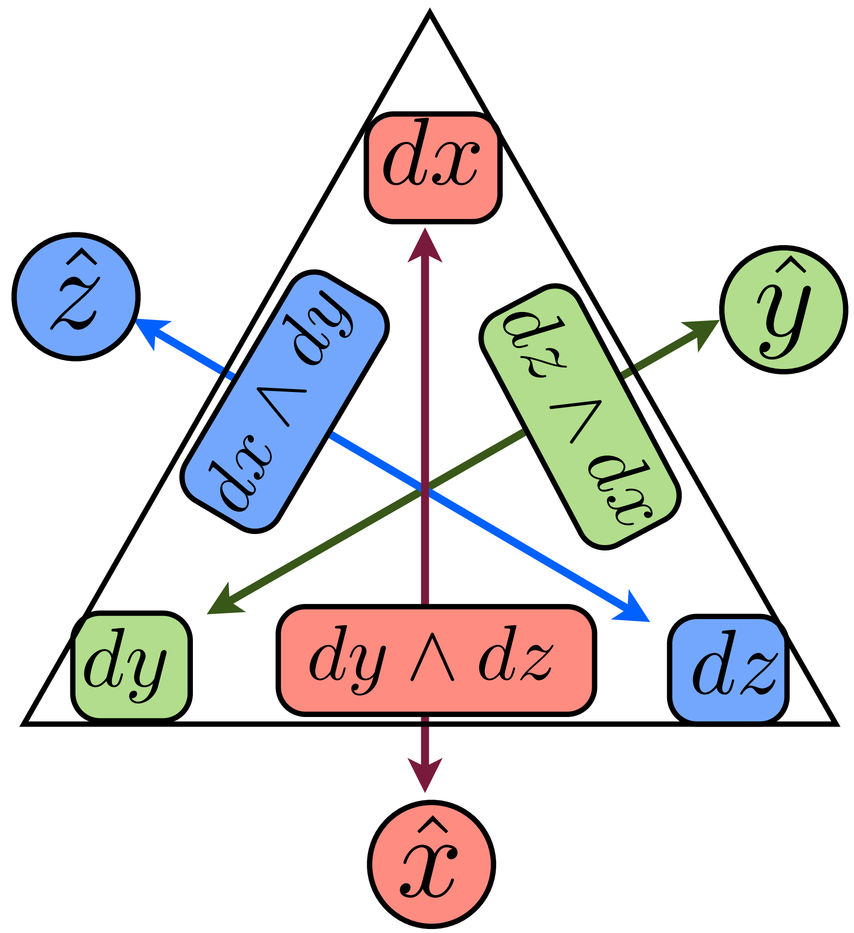
$$dx \wedge dy \wedge dz$$

3-Forms are Integrated over Volumes

$\int_V \alpha$ can be visualized as a number of cubes contained in the volume of integration.



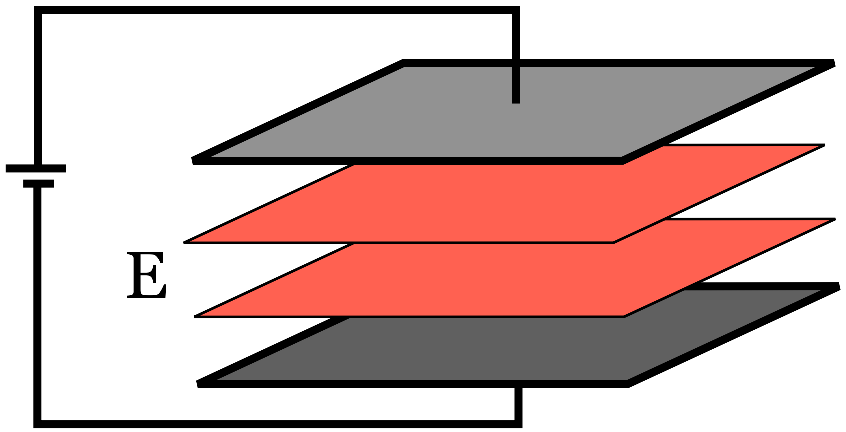
Duality Diagram



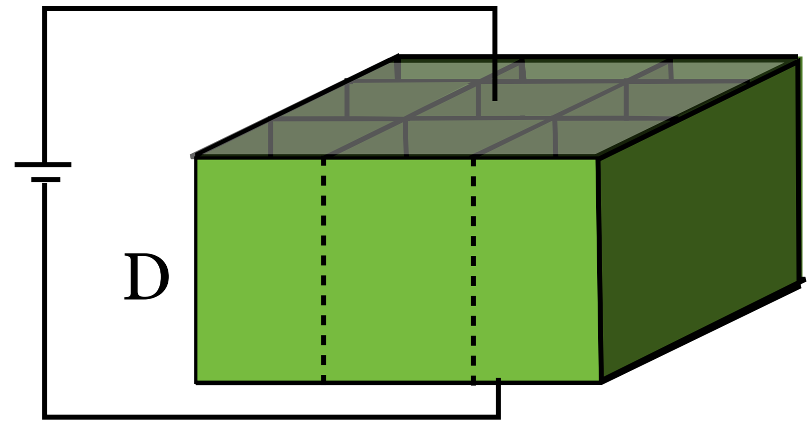
Scalar density: 1 $\longrightarrow$ $dx \wedge dy \wedge dz$

Scalar: 1 $\longrightarrow$ 0-form

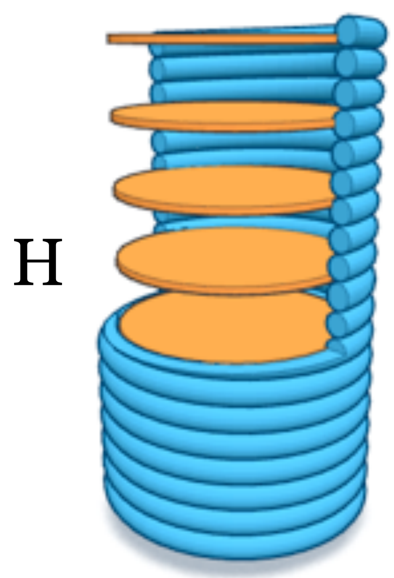
Electromagnetic Fields E, D, H, and B



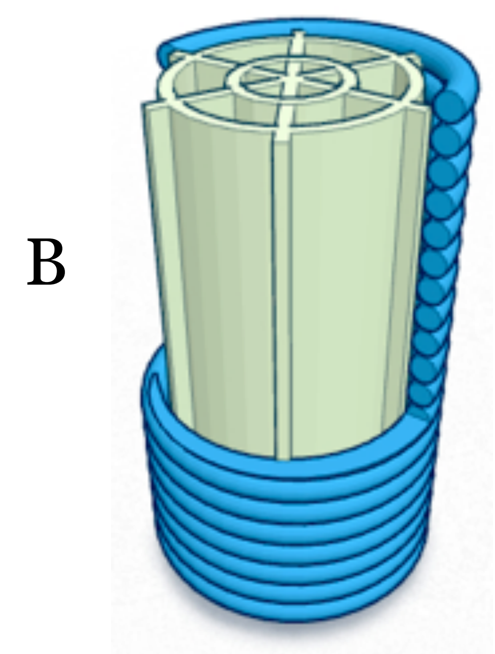
Electric field Intensity, 1-form



Electric Flux Density, 2-form



Magnetic Field Intensity, 1-form



Magnetic Flux Density, 2-form

Electromagnetic Fields E, D, H, and B

$$\oint_{\mathrm{C}} E = -\frac{d}{dt}\int_{A} B$$

$$\oint_{\mathrm{C}} H = \frac{d}{dt}\int_{A} D + \int_{A} J$$

$$\oint_{\mathrm{A}} D = \int_{V} \rho$$

$$\oint_{\mathrm{A}} B = 0.$$

Wedge Product

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$$

For two 1-forms, p=1 and q=1,
the product is antisymmetric

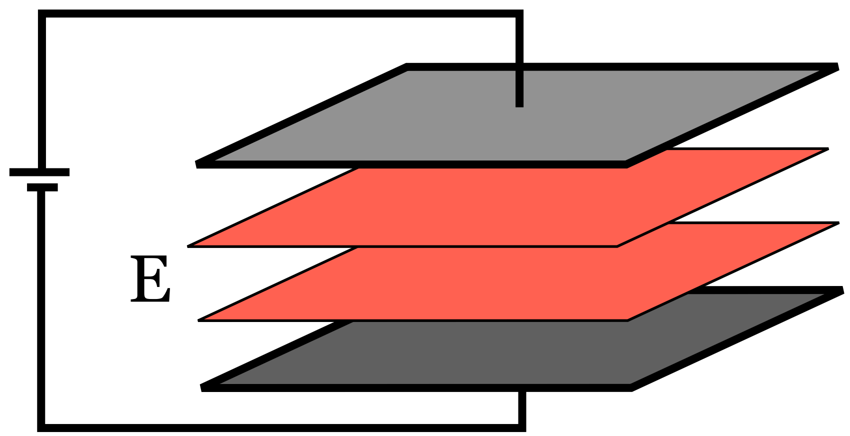
$$dx \wedge dx = -dx \wedge dx = 0$$

So,

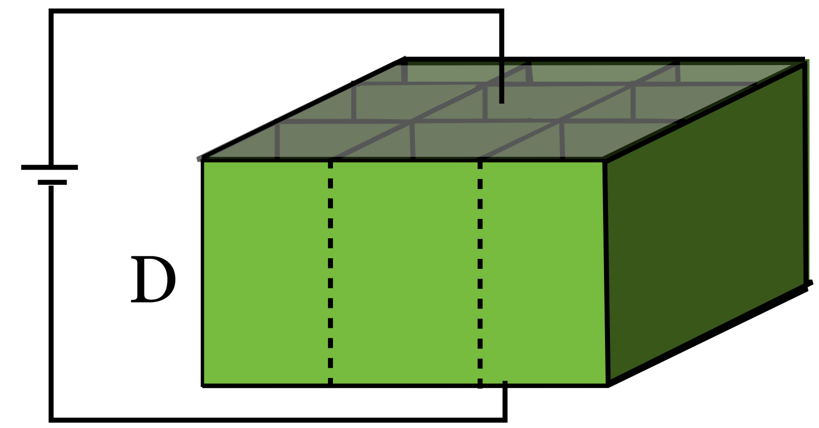
$$dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$$

$$dx \wedge dy = -dy \wedge dx$$, etc.

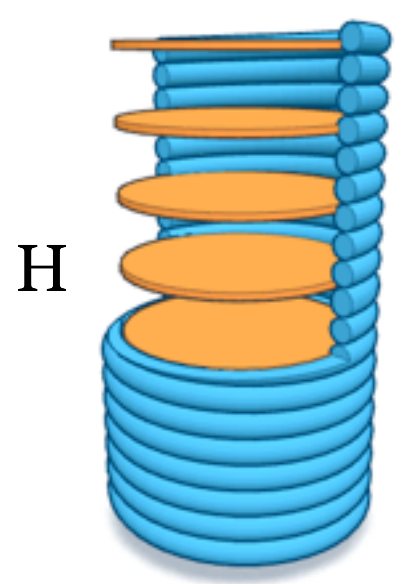
3D Printed Models



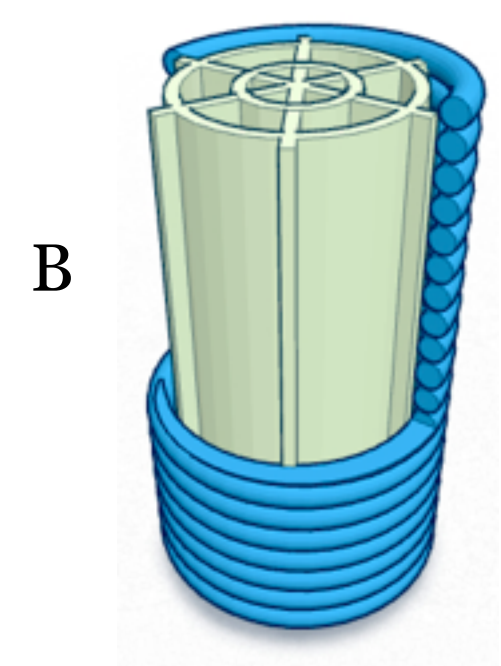
Electric field Intensity, 1-form



Electric Flux Density, 2-form



Magnetic Field Intensity, 1-form



Magnetic Induction, 2-form

Vector Identities

$$A \times (B \times C) = (C \times B) \times A = B(A \cdot C) - C(A \cdot B)$$

$$\nabla(uv) = u\nabla v + v\nabla u$$

$$\nabla(A \cdot B) = A \times (\nabla \times B) + (A \cdot \nabla)B + B \times (\nabla \times A) + (B \cdot \nabla)A$$

$$\nabla \cdot uA = u\nabla \cdot A + A \cdot \nabla u$$

$$\nabla \cdot (A \times B) = B \cdot \nabla \times A - A \cdot \nabla \times B$$

$$\nabla \times (uA) = u\nabla \times A - A \times \nabla u$$

$$\nabla \times (A \times B) = (B \cdot \nabla)A + A(\nabla \cdot B) - (A \cdot \nabla)B - B(\nabla \cdot A)$$

$$\overline{(\nabla \cdot A)B} = (A \cdot \nabla)B + B(\nabla \cdot A)$$

$$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - (\nabla \cdot \nabla)A$$

http://hyperphysics.phy-astr.gsu.edu/hbase/vecal2.html#c4

Differential Form Version of Vector Identities

Cross and dot product are equivalent to special cases of the **wedge product** which obeys

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$$

Div, Grad, and Curl are equivalent to the special cases of the **exterior derivative**, d.

$$dd\alpha = 0$$
$$d(\alpha \wedge \beta) = d\alpha + (-1)^p \alpha \wedge d\beta$$

Wedge Product

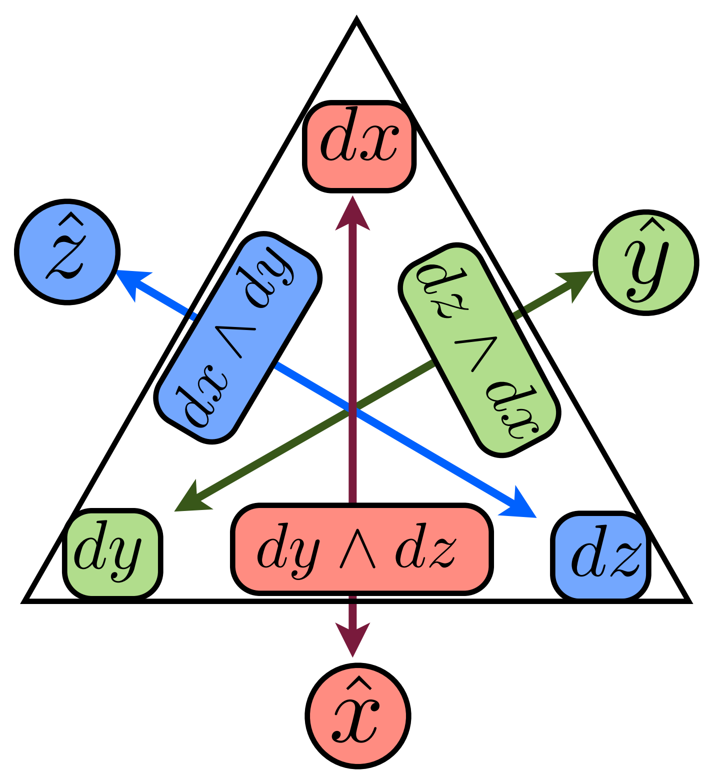
$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$$

The wedge of two 1-forms $\leftrightarrow$ Cross Product

$$A = 2\boxed{dx} \xrightarrow{\text{dual to}} \vec{A} = 2\boxed{\hat{x}}$$

$$B = \boxed{dy} \xrightarrow{\text{dual to}} \vec{B} = \boxed{\hat{y}}$$

$$A \wedge B = 2\boxed{dx \wedge dy} \xrightarrow{\text{dual to}} \vec{A} \times \vec{B} = 2\boxed{\hat{z}}$$



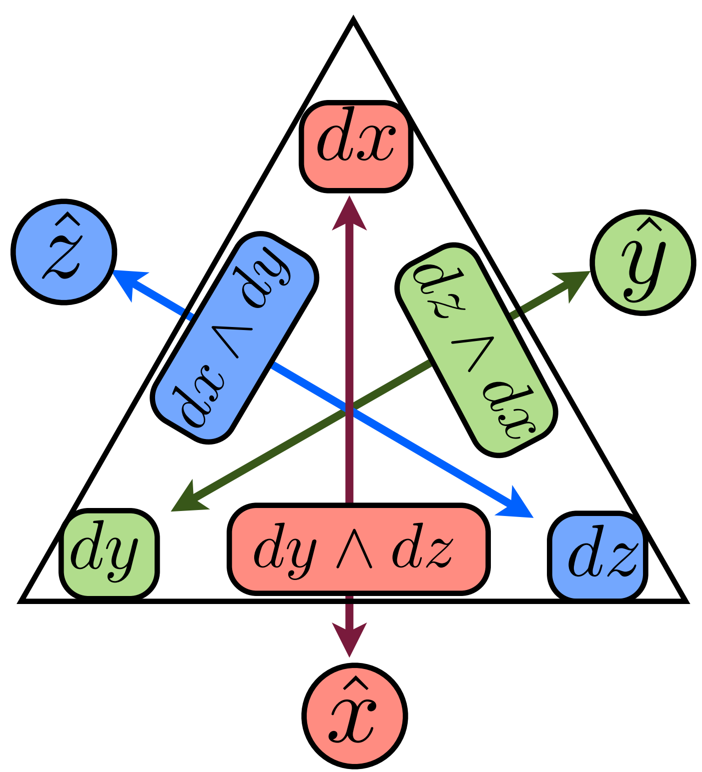
Wedge Product

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$$

The wedge of a 1-form and a 2-form $\longleftrightarrow$ Dot Product

$$A = 2\boxed{dx} \xrightarrow{\text{dual to}} \vec{A} = 2\boxed{\hat{x}}$$

$$C = \boxed{dy \wedge dz} \xrightarrow{\text{dual to}} \vec{C} = \boxed{\hat{x}}$$



Wedge Product

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$$

The wedge of a 1-form and a 2-form $\leftrightarrow$ Dot Product

$$A = 2\boxed{dx} \xrightarrow{\text{dual to}} \vec{A} = 2\boxed{\hat{x}}$$

$$C = \boxed{dy \wedge dz} \xrightarrow{\text{dual to}} \vec{C} = \boxed{\hat{x}}$$

$$A \wedge C = 2\boxed{dx \wedge dy \wedge dz} \xrightarrow{\text{dual to}} \vec{A} \cdot \vec{C} = 2$$

Scalar density: 1 $\longrightarrow$ $dx \wedge dy \wedge dz$

Dot and Cross Vector Product Identities

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

vectors

Dot and Cross Vector Product Identities

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

vectors

1-forms

$$A \wedge B \wedge C$$

Dot and Cross Vector Product Identities

$$\vec{A} \cdot \boxed{(\vec{B} \times \vec{C})} = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$A \wedge \boxed{B \wedge C}$$

The wedge of two 1-forms $\longleftrightarrow$ Cross Product

Dot and Cross Vector Product Identities

$$\boxed{\vec{A} \cdot \boxed{(\vec{B} \times \vec{C})}} = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$\boxed{A \wedge \boxed{B \wedge C}}$$

The wedge of two 1-forms $\longleftrightarrow$ Cross Product

The wedge of a 1-form and a 2-form $\longleftrightarrow$ Dot Product

Dot and Cross Vector Product Identities

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$A \wedge B \wedge C$$

Dot and Cross Vector Product Identities

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$A \wedge B \wedge C = -A \wedge C \wedge B$$

Swap B and C

Dot and Cross Vector Product Identities

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$A \wedge B \wedge C = -A \wedge C \wedge B = (-)(-)C \wedge A \wedge B$$

Swap C and A

Dot and Cross Vector Product Identities

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

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Dot and Cross Vector Product Identities

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot \boxed{(\vec{A} \times \vec{B})} = \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$A \wedge B \wedge C = -A \wedge C \wedge B = (-)(-)C \wedge A \wedge B = C \wedge \boxed{A \wedge B}$$

The wedge of two 1-forms $\longleftrightarrow$ Cross Product

Dot and Cross Vector Product Identities

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \boxed{\vec{C} \cdot \boxed{(\vec{A} \times \vec{B})}} = \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$A \wedge B \wedge C = -A \wedge C \wedge B = (-)(-) C \wedge A \wedge B = \boxed{C \wedge \boxed{A \wedge B}}$$

The wedge of two 1-forms $\longleftrightarrow$ Cross Product

The wedge of a 1-form and a 2-form $\longleftrightarrow$ Dot Product

Exterior Derivative

$$d \equiv \frac{d}{dx}dx + \frac{d}{dy}dy + \frac{d}{dz}dz$$

d of a 0-form ↔ gradient $\quad df \longrightarrow \vec{\nabla} f$

d of a 1-form ↔ curl $\quad dA \longrightarrow \vec{\nabla} \times \vec{A}$

d of a 2-form ↔ divergence $\quad dA \longrightarrow \vec{\nabla} \cdot \vec{A}$

d of a 3-form is zero

degree of the form α

$$dd\alpha = 0$$

$$d(\alpha \wedge \beta) = d\alpha + (-1)^{p} \alpha \wedge d\beta$$

Div, Grad, and Curl

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$$

$$\begin{aligned}
\overrightarrow{\mathbf{\nabla}} \cdot (\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}}) &= \frac{\partial}{\partial x}(A_y B_z - A_z B_y) + \frac{\partial}{\partial y}(A_z B_x - A_x B_z) + \frac{\partial}{\partial z}(A_x B_y - A_y B_x) \\
&= A_y \frac{\partial B_z}{\partial x} + B_z \frac{\partial A_y}{\partial x} - A_z \frac{\partial B_y}{\partial x} - B_y \frac{\partial A_z}{\partial x} + A_z \frac{\partial B_x}{\partial y} + B_x \frac{\partial A_z}{\partial y} - A_x \frac{\partial B_z}{\partial y} - B_z \frac{\partial A_x}{\partial y} \\
&\qquad + A_x \frac{\partial B_y}{\partial z} + B_y \frac{\partial A_x}{\partial z} - A_y \frac{\partial B_x}{\partial z} - B_x \frac{\partial A_y}{\partial z} \\
&= B_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + B_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + B_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - A_x \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) \\
&\qquad - A_y \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) - A_z \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = \overrightarrow{\mathbf{B}} \cdot (\overrightarrow{\mathbf{\nabla}} \times \overrightarrow{\mathbf{A}}) - \overrightarrow{\mathbf{A}} \cdot (\overrightarrow{\mathbf{\nabla}} \times \overrightarrow{\mathbf{B}}).
\end{aligned}$$

Div, Grad, and Curl

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

Example: α and β 1-forms

$$d(A \wedge B) = dA \wedge B + (-1)^1 A \wedge dB = dA \wedge B - A \wedge dB$$

1-forms $\leftrightarrow$ vectors

$\vec{A}$ $\vec{B}$

Div, Grad, and Curl

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

Example: α and β 1-forms

$$\boxed{d\left(\boxed{A \wedge B}\right)} = dA \wedge B + (-1)^1 A \wedge dB = dA \wedge B - A \wedge dB$$

The wedge of two 1-forms $\leftrightarrow$ Cross Product

d of a 0-form $\leftrightarrow$ gradient
d of a 1-form $\leftrightarrow$ curl
d of a 2-form $\leftrightarrow$ divergence
d of a 3-form is zero

$$\boxed{\vec{\nabla} \cdot (\vec{A} \boxed{\times} \vec{B})}$$

Div, Grad, and Curl

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

Example: α and β 1-forms

$$d(A \wedge B) = dA \wedge B + (-1)^1 A \wedge dB = \boxed{\boxed{dA} \wedge B} - A \wedge dB$$

The wedge of a 1-form and a 2-form ⟷ Dot Product

d of a 0-form ⟷ gradient
d of a 1-form ⟷ curl
d of a 2-form ⟷ divergence
d of a 3-form is zero

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \boxed{\vec{B} \cdot (\boxed{\vec{\nabla} \times \vec{A}})}$$

Div, Grad, and Curl

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

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$$d(A \wedge B) = dA \wedge B + (-1)^1 A \wedge dB = dA \wedge B - \boxed{A \wedge \boxed{dB}}$$

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$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \boxed{\vec{A} \cdot (\boxed{\vec{\nabla} \times \vec{B}})}$$

Div, Grad, and Curl

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$$d(A \wedge B) = dA \wedge B + (-1)^1 A \wedge dB = dA \wedge B - A \wedge dB$$

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$$\begin{aligned}
\vec{\boldsymbol{\nabla}} \cdot (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) &= \frac{\partial}{\partial x}(A_y B_z - A_z B_y) + \frac{\partial}{\partial y}(A_z B_x - A_x B_z) + \frac{\partial}{\partial z}(A_x B_y - A_y B_x) \\
&= A_y \frac{\partial B_z}{\partial x} + B_z \frac{\partial A_y}{\partial x} - A_z \frac{\partial B_y}{\partial x} - B_y \frac{\partial A_z}{\partial x} + A_z \frac{\partial B_x}{\partial y} + B_x \frac{\partial A_z}{\partial y} - A_x \frac{\partial B_z}{\partial y} - B_z \frac{\partial A_x}{\partial y} \\
&\quad + A_x \frac{\partial B_y}{\partial z} + B_y \frac{\partial A_x}{\partial z} - A_y \frac{\partial B_x}{\partial z} - B_x \frac{\partial A_y}{\partial z} \\
&= B_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + B_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + B_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - A_x \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) \\
&\quad - A_y \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) - A_z \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) = \vec{\mathbf{B}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}) - \vec{\mathbf{A}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}).
\end{aligned}$$

Goals

- Create an on-line repository of student-friendly learning materials for differential forms
    - Side-by-side derivations with vectors and differential forms
    - Detailed sample problems in both languages for easy comparison
    - Models for 3D printing
    - Applications to other subfields of physics (fluids, general relativity, etc.)
    - Applications to computation

Thank you!

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